Due Fri

1.2 – Gaussian Elimination

7. Solve the system by Gaussian elimination.

$$x - y + 2z - w = -1$$
$$2x + y - 2z - 2w = -2$$
$$-x + 2y - 4z + w = 1$$
$$3x - 3w = -3$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right]$$

1 2 3

$R_2 \to R_2 - 2R_1$

$$\begin{array}{ccccc} 2 & 1 & -2 & -2 & -2 \\ -2 & 2 & -4 & 2 & 2 \\ \hline 0 & 3 & -6 & 0 & 0 \end{array}$$

$R_3 \to R_3 + R_1$

$$\begin{array}{ccccc} -1 & 2 & -4 & 1 & 1 \\ 1 & -1 & 2 & -1 & -1 \\ \hline 0 & 1 & -2 & 0 & 0 \end{array}$$

$R_4 \to R_4 - 3R_1$

$$\begin{array}{ccccc} 3 & 0 & 0 & -3 & -3 \\ -3 & 3 & -6 & 3 & 3 \\ \hline 0 & 3 & -6 & 0 & 0 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right]$$

☺

$R_3 \to R_3 - \frac{1}{3}R_2$

$R_4 \to R_4 - R_2$

$R_2 \Leftrightarrow R_3$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

row echelon form via Gaussian elimination

We can solve the system by substituting:

$$x - y + 2z - w = -1$$

$$y - 2z = 0 \Rightarrow y = 2z$$

$$x - \cancel{2z} + \cancel{2z} - w = -1 \Rightarrow x = w - 1$$

Let $s = z$, $t = w$.

$$x = t - 1 \quad z = s$$
$$y = 2s \quad w = t$$

Row echelon form (#1-3 below) and **reduced row echelon form** (#1-4 below) of a matrix – to be of this form, a matrix must have the following properties:

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. This is a **leading 1.**
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column.

$$\left[\begin{array}{cccc|c} 1 & \boxed{-1} & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \Rightarrow R_1 + R_2$

$$\begin{array}{ccccc} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ \hline 1 & 0 & 0 & -1 & -1 \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc|c} 1 & \boxed{0} & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$\rightarrow x - w = -1 \Rightarrow x = w - 1$

$\rightarrow y - 2z = 0 \Rightarrow y = 2z$

w & z are free variables

This is reduced row echelon form (rref) via Gauss-Jordan elimination

leading 1s correspond to leading variables

columns that contain leading 1s are pivot columns

Leading variables correspond to the leading 1's in the augmented matrix for a linear system.

Free variables are the remaining variables in the linear system.

In a **homogeneous linear system**, the constant terms are all zero. That is, it has the form

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & 0 \end{array}$$

Theorem 1.2.1 Free Variable Theorem for Homogeneous Systems
If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

Theorem 1.2.2 A homogeneous linear system with more unknowns than equations has infinitely many solutions.

pf: A homogeneous system always has at least the trivial solution.
The rows of an augmented matrix represent equations and the columns that comprise coefficients represent unknowns. If a matrix has fewer rows than columns, then it has fewer leading 1s than variables. Such a system has at least one free variable and so has infinitely many solutions.

28. What condition, if any, must a, b, and c satisfy for the linear system to be consistent?

$$x + 3y + z = a$$
$$-x - 2y + z = b$$
$$3x + 7y - z = c$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ -1 & -2 & 1 & b \\ 3 & 7 & -1 & c \end{array}\right]$$

1 2

$R_2 \to R_1 + R_2$

$$\begin{array}{cccc} 1 & 3 & 1 & a \\ -1 & -2 & 1 & b \\ \hline 0 & 1 & 2 & a+b \end{array}$$

$R_3 \to R_3 - 3R_1$

$$\begin{array}{cccc} 3 & 7 & -1 & c \\ -3 & -9 & -3 & -3a \\ \hline 0 & -2 & -4 & c-3a \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & -2 & -4 & c-3a \end{array}\right]$$

$R_3 \to R_3 + 2R_2$

$$\begin{array}{cccc} 0 & -2 & -4 & c-3a \\ 0 & 2 & 4 & 2b+2a \\ \hline 0 & 0 & 0 & c+2b-a \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & 0 & 0 & c+2b-a \end{array}\right] \to 0 = c + 2b - a$$

The system is consistent if $a = 2b + c$.

Solve the given linear system ~~by any method~~.

using an augmented matrix

16.

$$\begin{aligned} 2x - y - 3z &= 0 \\ -x + 2y - 3z &= 0 \\ x + y + 4z &= 0 \end{aligned}$$

$$\begin{array}{cccc} 1 & 2 & 4 & 3 \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & \boxed{-1} & \boxed{-3} & 0 \\ \boxed{-1} & 2 & \boxed{-3} & 0 \\ \boxed{1} & \boxed{1} & 4 & 0 \end{array}\right]$$

writing these zeros is optional in a homogeneous system

$R_2 \to R_1 + 2R_2$ *

Not an elementary row operation

$$\begin{array}{ccc} 2 & -1 & -3 \\ -2 & 4 & -6 \\ \hline 0 & 3 & -9 \end{array}$$

Then $R_2 \to \frac{1}{3}R_2$

$R_3 \to R_3 + R_2$

$$\begin{array}{ccc} 1 & 1 & 4 \\ -1 & 2 & -3 \\ \hline 0 & 3 & 1 \end{array}$$

$$\begin{bmatrix} 2 & -1 & -3 \\ 0 & 1 & -3 \\ 0 & 3 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{array}{ccc} 0 & 3 & 1 \\ 0 & -3 & 9 \\ \hline 0 & 0 & 10 \end{array}$$

then $R_3 \to \frac{1}{10}R_3$

$$\begin{bmatrix} 2 & -1 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1$

$$\begin{array}{ccc} 0 & 1 & -3 \\ 0 & 0 & 3 \\ \hline 0 & 1 & 0 \end{array}$$

$R_1 \to R_1 + 3R_3$

$$\begin{array}{ccc} 2 & -1 & -3 \\ 0 & 0 & 3 \\ \hline 2 & -1 & 0 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 + R_2$

then

$R_1 \to \frac{1}{2}R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\boxed{(x, y, z) = (0, 0, 0)}$$

this is the <u>trivial solution</u>.

19.

$$\begin{aligned} 2x + 2y + 4z &= 0 \\ w - y - 3z &= 0 \\ 2w + 3x + y + z &= 0 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$$

$$\begin{array}{cccc} w & x & y & z \\ & 6 & 5 & 4 \end{array}$$

$$\begin{bmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & -1 & -3 \\ 2 & 3 & 1 & 1 \\ -2 & 1 & 3 & -2 \end{bmatrix}$$

$$\begin{array}{cccc} 1 & 2 & 3 & \end{array}$$

$R_2 \leftrightarrow R_1$

Then

$R_2 \to \frac{1}{2} R_2$

$$\begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 2 & 3 & 1 & 1 \\ -2 & 1 & 3 & -2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\begin{array}{rrrr} 2 & 3 & 1 & 1 \\ -2 & 0 & 2 & 6 \\ \hline 0 & 3 & 3 & 7 \end{array}$$

$R_4 \to R_4 + R_3$

$$\begin{array}{rrrr} -2 & 1 & 3 & -2 \\ 2 & 3 & 1 & 1 \\ \hline 0 & 4 & 4 & -1 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 3 & 7 \\ 0 & 4 & 4 & -1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{array}{rrrr} 0 & 3 & 3 & 7 \\ 0 & -3 & -3 & -6 \\ \hline 0 & 0 & 0 & 1 \end{array}$$

$R_4 \to R_4 - 4R_2$

$$\begin{array}{rrrr} 0 & 4 & 4 & -1 \\ 0 & -4 & -4 & -8 \\ \hline 0 & 0 & 0 & -9 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -9 \end{bmatrix}$$

$R_4 \to -\frac{1}{9} R_4$

$$\to \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_4 \to R_3 - R_4$

$$\begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_3$

$$\begin{array}{rrrr} 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -2 \\ \hline 0 & 1 & 1 & 0 \end{array}$$

$R_1 \to R_1 + 3R_3$

$$\begin{array}{rrrr} 1 & 0 & -1 & -3 \\ 0 & 0 & 0 & 3 \\ \hline 1 & 0 & -1 & 0 \end{array}$$

$$\begin{array}{cccc} w & x & y & z \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$w - y = 0$
$x + y = 0$
$z = 0$

Let $t = y$

$$\boxed{\begin{aligned} w &= t \\ x &= -t \\ y &= t \\ z &= 0 \end{aligned}}$$

23. The augmented matrix for a linear system is given in which the asterisk represents an unspecified real number. Determine whether the system is consistent, and if so whether the solution is unique. Answer "inconclusive" if there is not enough information to make a decision.

a. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$ consistent, unique

b. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right]$ consistent, infinitely many solutions

c. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{array}\right] \to 0 = 1 \Rightarrow$ inconsistent system

d. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 1 & * \end{array}\right]$

If we have
0 0 0 0
0 0 1 0,
then $z = 0$

If we have
0 0 1 0
0 0 1 1
inconsistent

inconclusive